座標の表し方と▽の極座標変換

・座標の表し方

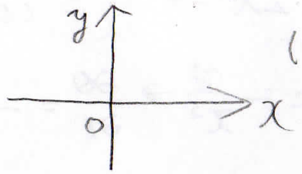
☆ 一次元

独立なパラメータ

(x)

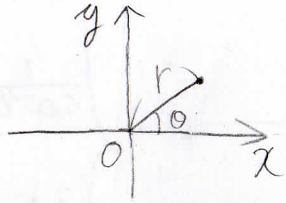
☆ 二次元

直交座標



(x, y)

極座標

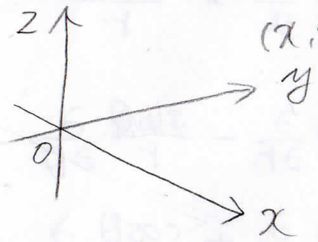


(r, θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

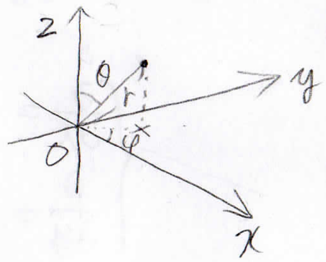
☆ 三次元

直交座標



(x, y, z)

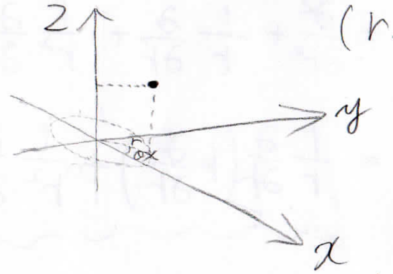
極座標



(r, θ, φ)

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

円筒座標



(r, θ, z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z \end{cases}$$

・$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ を極座標に変換

☆ 偏微分の順番入れ替え

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial^2}{\partial y \partial x} f(x, y)$$

常には成り立たない（$f(x, y)$ が C^2 級でないといけない）

但し、科学で扱う実用的な関数においては成り立つことが多い

☆ 実際に変形

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \rightarrow \begin{cases} r^2 = x^2 + y^2 + z^2 \\ \cos\theta = \dfrac{z}{r} \\ \tan\varphi = \dfrac{y}{x} \end{cases}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}$$

① $2r\dfrac{\partial r}{\partial x} = 2x \rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \sin\theta\cos\varphi$

② $-\sin\theta\dfrac{\partial \theta}{\partial x} = -\dfrac{zx}{r^3} \rightarrow \dfrac{\partial \theta}{\partial x} = \dfrac{\cos\theta\cos\varphi}{r}$

③ $\dfrac{1}{\cos^2\varphi}\dfrac{\partial \varphi}{\partial x} = -\dfrac{y}{x^2} \rightarrow \dfrac{\partial \varphi}{\partial x} = -\dfrac{\sin\varphi}{r\sin\theta}$

同様にすると

$$\frac{\partial r}{\partial y} = \sin\theta\sin\varphi, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\varphi}{r}, \quad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}$$

$$\frac{\partial r}{\partial z} = \cos\theta, \quad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r}, \quad \frac{\partial \varphi}{\partial z} = 0$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial \psi}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial \psi}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial \varphi} \qquad \left(\{f(x)g(x)\}' = f'(x)g(x) + f(x)g(x)\right)$$

$$= \sin\theta\cos\varphi\frac{\partial \psi}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial \psi}{\partial \theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial \psi}{\partial \varphi}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \sin^2\theta\cos^2\varphi\frac{\partial^2 \psi}{\partial r^2} - \frac{\sin\theta\cos\theta\cos^2\varphi}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{\sin\theta\cos\theta\cos^2\varphi}{r}\frac{\partial^2 \psi}{\partial r\partial \theta} + \frac{\sin\varphi\cos\varphi}{r^2}\frac{\partial \psi}{\partial \varphi}$$

$$- \frac{\sin\varphi\cos\varphi}{r}\frac{\partial^2 \psi}{\partial r\partial \varphi} + \frac{\cos^2\theta\cos^2\varphi}{r}\frac{\partial \psi}{\partial r} + \frac{\sin\theta\cos\theta\cos^2\varphi}{r}\frac{\partial^2 \psi}{\partial \theta\partial r}$$

$$- \frac{\sin\theta\cos\theta\cos^2\varphi}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{\cos^2\theta\cos^2\varphi}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cos^2\theta\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial \psi}{\partial \varphi}$$

$$- \frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial^2 \psi}{\partial \theta\partial \varphi} + \frac{\sin^2\varphi}{r}\frac{\partial \psi}{\partial r} - \frac{\sin\varphi\cos\varphi}{r}\frac{\partial^2 \psi}{\partial \varphi\partial r}$$

$$+ \frac{\cos\theta\sin^2\varphi}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} - \frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial^2 \psi}{\partial \varphi\partial \theta} + \frac{\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial \psi}{\partial \varphi} + \frac{\sin^2\varphi}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \cdots$$

$$\frac{\partial^2 \psi}{\partial z^2} = \cdots$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

・練習問題

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ を極座標で表すと?

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \rightarrow \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

答え

$$\begin{cases} 2r\frac{\partial r}{\partial x} = 2x \rightarrow \frac{\partial r}{\partial x} = \cos\theta \\ \frac{1}{\cos^2\theta}\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \rightarrow \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \\ 2r\frac{\partial r}{\partial y} = 2y \rightarrow \frac{\partial r}{\partial y} = \sin\theta \\ \frac{1}{\cos^2\theta}\frac{\partial \theta}{\partial y} = \frac{1}{x} \rightarrow \frac{\cos\theta}{r} \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{cases}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial \psi}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial \psi}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial \varphi}$$

$$= \sin\theta \sin\varphi \frac{\partial \psi}{\partial r} + \frac{\cos\theta \sin\varphi}{r}\frac{\partial \psi}{\partial \theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial \psi}{\partial \varphi}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \sin^2\theta \sin^2\varphi \frac{\partial^2 \psi}{\partial r^2} - \frac{\sin\theta\cos\theta\sin^2\varphi}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{\sin\theta\cos\theta\sin^2\varphi}{r}\frac{\partial^2 \psi}{\partial r\partial \theta} - \frac{\sin\varphi\cos\varphi}{r^2}\frac{\partial \psi}{\partial \varphi}$$

$$+ \frac{\sin\varphi\cos\varphi}{r}\frac{\partial^2 \psi}{\partial r\partial \varphi} + \frac{\cos^2\theta\sin^2\varphi}{r}\frac{\partial \psi}{\partial r} + \frac{\sin\theta\cos\theta\sin^2\varphi}{r}\frac{\partial^2 \psi}{\partial \theta\partial r} - \frac{\sin\theta\cos\theta\sin^2\varphi}{r^2}\frac{\partial \psi}{\partial \theta}$$

$$+ \frac{\cos^2\theta\sin^2\varphi}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} - \frac{\cos^2\theta\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial \psi}{\partial \varphi} + \frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial^2 \psi}{\partial \theta\partial \varphi} + \frac{\cos^2\varphi}{r}\frac{\partial \psi}{\partial r}$$

$$+ \frac{\sin\varphi\cos\varphi}{r}\frac{\partial^2 \psi}{\partial \varphi\partial r} + \frac{\cos\theta\cos^2\varphi}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} + \frac{\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta}\frac{\partial^2 \psi}{\partial \varphi\partial \theta} - \frac{\sin\varphi\cos\varphi}{r^2\sin^2\theta}\frac{\partial \psi}{\partial \varphi} + \frac{\cos^2\varphi}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \varphi^2}$$

$$\frac{\partial \psi}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial \psi}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial \psi}{\partial \theta} + \frac{\partial \varphi}{\partial z}\frac{\partial \psi}{\partial \varphi}$$

$$= \cos\theta\frac{\partial \psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \psi}{\partial \theta}$$

$$\frac{\partial^2 \psi}{\partial z^2} = \cos^2\theta\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial \psi}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 \psi}{\partial r\partial \theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial \psi}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 \psi}{\partial \theta\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial \psi}{\partial \theta}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$$